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A GENERAL THEORY OF WIDEBAND MATCHING
WITH DISSIPATIVE 4-POLES

by

R. La Rosa and H. J. Carlin

Research Report R-308-53, PIB-247

for

OFFICE OF NAVAL RESEARCH

Contract NOnr-292(00)

Project Designation 075-215

February 17, 1953



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Acknowledgment
Abstract
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ABSTRACT

An arbitrary load impedance is matched over a wide band of frequencies to a constant-resistance generator by means of a passive, dissipative 4-pole. Theoretical limitations are derived which relate the power dissipated in the load to the impedance mismatch seen by the generator. The optimum matching network from the standpoint of maximum power transmission efficiency is a 4-pole which requires no more than one resistor in its synthesis.

The theory can also be applied to the wide-band matching of an arbitrary generator to a constant-resistance load. Normalization methods are described which can take account of the variation with frequency of the available power of the generator.

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1. Statement of the Problem

The load impedance $Z(p)$ is given as a function of the complex frequency variable $p = \sigma + j\omega$ where σ is the negative of the damping constant and ω is the radian frequency of an exciting wave. The $j\omega$ axis of the p -plane corresponds to steady-state sinusoidal waves.

The matching network of Fig. MRI-13039a is to transmit power from the unit internal resistance generator to the load and also to limit the impedance mismatch at the generator terminals. The impedance mismatch is specified by a voltage reflection coefficient

$$s_{11} = \frac{z_{in} - 1}{z_{in} + 1} \quad (1)$$

which is a function of the complex variable p . z_{in} is the impedance seen looking to the right from the generator terminals.

In addition to the specification of load impedance, two quantities are presumed to be of engineering importance. These are the magnitude $|s_{11}(j\omega)|$ of the input reflection coefficient and the power $P(\omega)$ which reaches the load at steady-state sinusoidal frequencies.

Lossless matching networks are desirable in many problems because they permit all the power, except that which is reflected back to the generator, to be transmitted to the load. That is, for a lossless matching network

$$P(\omega) = 1 - |s_{11}(j\omega)|^2 \quad (2)$$

and if the reflection coefficient is kept small over the frequency range of interest, $P(\omega)$ will be nearly the total available power of the generator which, in the case of Fig. MRI-13039a is one watt.

Over any given frequency range $|s_{11}(j\omega)|$ cannot be made arbitrarily small by means of a lossless matching network. The limitations on the performance of lossless matching networks were first investigated by Bode (Reference 1) for simple load impedances. Fano (References 2,3) later extended the analysis to the most general load impedance.

There are at least three reasons why lossy matching networks may be better than lossless matching networks in specific applications. First, a lossless matching network may not be able to provide the desired small $|s_{11}(j\omega)|$ over the required frequency range. Second, $P(\omega)$ and $|s_{11}(j\omega)|$

are not independently controlled when a lossless matching network is used. This means, for example, that if $P(\omega)$ is to be decreased in some frequency range for attenuation equalization purposes, the reflection coefficient $S_{11}(j\omega)$ must automatically increase in magnitude in accordance with equation (2). Third, a dissipative matching network might have a simpler form than a lossless network. In some cases, the increased simplicity is an advantage which offsets a slight waste of power in the lossy matching network.

Theoretical limitations will be determined which relate $|S_{11}(j\omega)|$ and $P(\omega)$ to the given load impedance when the matching network is composed of lumped inductors, capacitors, resistors, and ideal transformers connected in any fashion without regard to complexity. The use of the scattering matrix description must be introduced in order to accomplish this aim.

2. Scattering Matrix Realizability

A network is here defined as a system of lumped inductors, capacitors, resistors, and ideal transformers, n pairs of terminals are provided for the purposes of connecting this system to other networks. Each pair of terminals is called a "port". A network with n ports is called an "n-port". If the network does not contain any resistors it is a lossless n-port. If the network is not prevented from containing resistors it is a lossy n-port. The word passive is consistently omitted because any device containing a source of energy will be called a generator.

The open-circuit impedance matrix $[Z]$ is an $n \times n$ matrix which relates the voltages at all the ports to the currents which flow in the accessible terminal pairs defining the ports. The short-circuit admittance matrix $[Y]$ is also $n \times n$ and gives the currents at the ports in terms of the voltages. Impedance and admittance descriptions are useful in network analysis and synthesis but for the purposes of this paper another description is essential: the scattering matrix $[S]$.

If polarities of voltage and current are chosen as in Fig. MRI-13039b, the following matrix relations are true where $[I]$ is the identity matrix of order n

$$[Z] = [Y]^{-1} \quad (3)$$

$$[S] = [Z]^{-1} [Z+I]^{-1} = [I-Y] [I+Y]^{-1} \quad (4)$$

$$[Z] = [I-S]^{-1} [I+S] \quad (5)$$

$$[Y] = [I+S]^{-1} [I-S] \quad (6)$$

The scattering matrix has a physical interpretation in terms of incident and reflected voltage waves. The main diagonal element S_{kk} is the voltage reflection coefficient at port k referred to a one ohm source when all other ports except k are terminated in one ohm. That is, if the impedance looking into port k is Z_k when all other ports are terminated in one ohm,

$$S_{kk} = \frac{Z_k - 1}{Z_k + 1} \quad (7)$$

The off-diagonal elements of $[S]$ are transmission coefficients. For example, S_{jk} is the voltage of port j when a one watt, one ohm generator is connected to port k and all other ports are terminated in one ohm. This is illustrated in Fig. MRI-13039c. It should be noted that this definition of $[S]$ depends on the existence of $[Z]$ or $[Y]$. A more general formulation is possible which does not involve either of these matrices. This is important since for many physical networks $[Z]$ or $[Y]$ may not exist, but $[S]$ always exists.

Two papers by Belevitch (References 4,5) give the realizability requirements for a scattering matrix and these are stated below as theorems.

Theorem 1

The necessary and sufficient conditions for $[S]$ to correspond to a physically realizable n -port are:

- (a) $[S]$ is symmetrical and its elements are rational functions of $p = \sigma + j\omega$ and are real for $\omega = 0$.
- (b) Elements of $[S]$ have no poles in the interior of the right hand half of the p -plane, i.e. the region^a where $\sigma > 0$.
- (c) $[1 - S^*(j\omega) S(j\omega)]$ is the matrix of a positive-definite or positive-semi-definite hermitian form for $-\infty < \omega < \infty$.
[The asterisk denotes the complex conjugate, $S^*(j\omega) = S(-j\omega)$]

Theorem 2

The necessary and sufficient conditions for $|S|$ to correspond to a physically realizable lossless n -port are:

- (a) $[S]$ is symmetrical and its elements are rational functions in $p = \sigma + \omega j$ and are real for $\omega = 0$.

^a(The original Belevitch statement (Reference 4) includes the $\sigma = 0$ boundary but this is unnecessary since it is insured by parts (a) and (c)).

(b) Elements of $[S]$ have no poles in the right hand half of the p -plane.

(c) $[S]$ is unitary for $\epsilon = 0$.^b That is, $[S^T(j\omega) S(j\omega)] = [1]$.

3. Darlington Representation of $Z(p)$

Any realizable impedance $Z(p)$ may be obtained as the input impedance of a lossless 2-port terminated in one ohm. This idea, shown by Darlington (Reference 6), is very useful because it permits the load impedance $Z(p)$ of Fig. MRI-13039a to be transformed into the lossless 2-port E of Fig. MRI-13039d terminated in one ohm. The matching 2-port D and the lossless 2-port E in tandem can then be treated as a single 2-port S whose scattering matrix $[S]$ contains the relevant quantities $|S_{11}(j\omega)|$ which measures the degree of input mismatch and $|S_{12}(j\omega)|$ which is equal to $P(\omega)$.

The load impedance $Z(p)$ immediately specifies the input reflection coefficient $E_{11}(p)$ of the lossless 2-port E according to

$$E_{11}(p) = \frac{Z(p) - 1}{Z(p) + 1} \quad (8)$$

The lossless 2-port E is uniquely specified by $Z(p)$ except for the possible addition of all-pass phase-shifting networks in tandem with end 2 of the minimum network. The minimum network has an $E_{12}(p)$ containing the smallest number of right-hand zeros. The additional phase-shift networks have the effect of putting additional right-hand zeros in $E_{12}(p)$ and $E_{22}(p)$ without changing the magnitudes of these functions along the $p = j\omega$ axis, because these additional right half-plane zeros are always paired with corresponding zeros in the left half-plane.

In the matching problem, the extra phase-shifting sections are undesirable because they complicate the problem with useless information that eventually drops out of the calculations for $P(\omega)$ and $|S_{11}(j\omega)|$.

A complete discussion of the derivation of matrix $[E]$ from the specified $Z(p)$ is included in Reference 7.

^bThe subscript T denoting the transpose of $[S^*(j\omega)]$ is not necessary because of the symmetry of the scattering matrix i.e. $[S^*(j\omega) S(j\omega)] = [1]$.

4. Restrictions on the Overall Scattering Matrix [S]

The given $Z(p)$ has been changed into a lossless 2-port terminated in one ohm by means of the theory described in Section 3. This lossless 2-port in tandem with the matching network D forms a 2-port S whose input end is connected to a one ohm generator and whose output port is connected to a one ohm load. This is an ideal system for analysis by means of the scattering matrix description.

The elements of the scattering matrix $[S]$ of the overall network are related to the elements of scattering matrices $[D]$ and $[E]$ by the following equations:

$$S_{11} = D_{11} + \frac{D_{12} E_{11}}{1 - D_{22} E_{11}} \quad (9)$$

$$S_{12} = \frac{D_{12} E_{12}}{1 - D_{22} E_{11}} \quad (10)$$

$$S_{22} = E_{22} + \frac{E_{12} D_{22}}{1 - D_{22} E_{11}} \quad (11)$$

As stated before, $|S_{11}(j\omega)|$ and $|S_{12}(j\omega)|^2 = P(\omega)$ are quantities of engineering interest and the desired solution to the matching problem is a relation between these two functions and the given load impedance. The properties of the load which are relevant to the matched performance are more easily seen from the scattering matrix $[E]$ than from the load impedance $Z(p)$. The significant properties, as first shown by Fano (References 2,3) are the right-hand and boundary zeros of E_{12} and certain coefficients in the Taylor expansion for E_{22} about each of these zeros. The intuitive physical reason for this is that the passive matching network D cannot act as a power amplifier for steady-state sine waves or exponentially increasing waves of any frequency. Therefore, if 2-port E will not transmit such a wave, the tandem combination of D and E will not transmit the wave. Moreover, at such right-hand and boundary zeros of E_{12} , the matching network D cannot be "seen" from the output end of S because E is "opaque". Thus, right-hand and boundary zeros of E_{12} appear in S_{12} and certain coefficients in the S_{22} expansions about these zeros are independent of the elements of $[D]$ and are equal to the corresponding coefficients in the expansion for E_{22} .

The formal, mathematical restrictions on $[S]$ come from two sources: (1.) the overall network S must be realizable and (2.) the matching network D must be realizable as a separate entity. The first group of restrictions is independent of the load, while the second group is intimately connected

with the aforementioned significant properties of the load. The restrictions of groups 1 and 2 come from applying Theorem 1 to $[S]$ and $[D]$ respectively.

Satisfaction of Part (a) of either realizability theorem is taken for granted in what follows because all matrices are assumed to be symmetrical and their elements are real for real values of the complex frequency variable p . Nothing is done which is inconsistent with this assumption. All functions considered are rational functions.

Applying Theorem 1 to $[S]$, Part (b) requires that elements of $[S]$ have no poles in the interior of the right-hand half of the p -plane.

Part (c) requires that

$$1 - |s_{11}|^2 - |s_{12}|^2 \geq 0 \quad (p = j\omega) \quad (12)$$

$$1 - |s_{22}|^2 - |s_{12}|^2 \geq 0 \quad (p = j\omega) \quad (13)$$

$$\begin{aligned} (1 - |s_{12}|^2)^2 - |s_{22}|^2 - |s_{11}|^2 + |s_{11}s_{22}|^2 \\ - s_{11}s_{22}s_{12}^{*2} - s_{11}^{*}s_{22}^{*}s_{12}^2 \geq 0 \quad (p = j\omega) \end{aligned} \quad (14)$$

The left-hand sides of (12), (13) are the two principal diagonal elements and (14) is the determinant, of $[1 - S^*(j\omega) S(j\omega)]$. Either (12) or (13) may be disregarded because positiveness of one main diagonal element and the determinant insure positiveness of the other diagonal element.

The above statements are necessary and sufficient to insure that $[S]$ is the matrix of a physically realizable 2-port. The second group of requirements on $[S]$ will be found by applying Theorem 1 to matrix $[D]$. These requirements will insure that network S is composed of two realizable networks in tandem, one of which is the lossless 2-port specified by the given load impedance $Z(p)$.

Part (a) of Theorem 1 is taken for granted as before and Part (c) is considered next because it has already been satisfied by the first group of requirements on $[S]$.

Section 5.16 of page 148 of Reference 8 explains that $[1 - S^*(j\omega) S(j\omega)]$ is the matrix of the hermitian form which gives the power entering 2-port S in terms of the incident voltage waves at ports 1 and 2. Positive definiteness or semi-definiteness of this matrix for all ω insures that for any combination of magnitudes and phases of waves incident on ports 1 and 2,

the power entering S is always positive. The matrix $[1 - S^*(j\omega) S(j\omega)]$ may be identically zero for all ω if the matching network is lossless, but in the general case of a lossy matching network the determinant of the matrix and all its principal minors are greater than zero except for discrete values of ω .

The matrix $[E(j\omega)]$ is unitary for all values of ω . This means that lossless 2-port E neither absorbs nor produces power no matter what combination of incident amplitudes and phases is applied to its two ports at any frequency.

None of the power dissipated by S is absorbed in E , nor can any be generated. Therefore, matching 2-port D is the only power absorbing structure of the D , E tandem combination at all frequencies and for any combination of amplitudes and phases of incident voltage waves on the accessible ports of the system, i.e., the ports of S . Port 2 of D is not accessible when D and E are connected in tandem and treated as a single unit. Except at boundary zeros of E_{12} any phase and amplitude of incident voltage wave at port 2 of D may be obtained by applying a suitable wave to port 2 of S . By this means, waves incident on ports 1 and 2 of D may be independently controlled.

At boundary zeros of E_{12} , however, the incident wave on port 2 of D comes from the wave which passes through D and is reflected back into port 2 of D by network E which will not pass that particular frequency. Hence, at these isolated frequencies, the voltage waves incident on both ports of D are not subject to independent control. This lack of independent control exists only at discrete, isolated frequencies. If the matrix $[1 - D^*(j\omega) D(j\omega)]$ is positive-definite or semi-definite in the neighborhood of these $E_{12}(j\omega)$ zeros, the matrix will retain these properties at the frequency of the zero because all its elements are continuous functions.

It is now known that network D absorbs power for all frequencies and all amplitudes and phases of voltage waves incident on its two ports. This means that 2-port D satisfies the requirement of Theorem 1 Part (c), namely, $[1 - D^*(j\omega) D(j\omega)]$ is the matrix of a positive-definite or semi-definite hermitian form for all ω , when the sufficient condition that $[1 - S^*(j\omega) S(j\omega)]$ be positive or semi-definite is satisfied. Further this reasoning is easily inverted starting with the hypothesis that $[1 - D^*(j\omega) D(j\omega)]$ is positive or semi-definite. This means that D can only absorb power and since E is lossless the result is that $[1 - S^*(j\omega) S(j\omega)]$ is positive or semi-definite.

Thus, the necessary and sufficient conditions on $[S]$ have been determined which insure that $[D]$ satisfies Parts (a) and (c) of Theorem 1 and it remains only to find the requirements on $[S]$ which insure that $[D]$ satisfies Part (b).

Fano has shown (References 2,3) that if D_{22} corresponds to any realizable impedance according to

$$Z_2 = \frac{1 + D_{22}}{1 - D_{22}} \quad (15)$$

then certain restrictions are imposed on $S_{22}(p)$. These may be determined by examining equations 10 and 11. At each boundary and right-hand zero of E_{12} , $S_{22}(p)$ and $E_{22}(p)$ are expanded in Taylor series. If a particular right-hand zero is of order n , then corresponding coefficients in the two series are equal up to and including those of $(p - p_0)^{2n-1}$. In the case of a boundary zero, corresponding coefficients are equal up to and including those of $(p - j\omega_0)^{2n-2}$ or $(1/p)^{2n-2}$ in the case of a zero at infinity. The coefficients of the $(2n-1)$ power of the variable may be equal or they may differ in a certain, prescribed way. If they differ, the difference in the coefficient is such as to cause the vector representing $S_{22}(j\omega)$ to have a slower clockwise rotation than $E_{22}(j\omega)$ as ω is increased in the neighborhood of the E_{12} zero. The $S_{22}(j\omega)$ locus and the $E_{22}(j\omega)$ vectors must always rotate clockwise at a point of tangency to cause the right-hand half of the p -plane to map inside of the unit circle in the S_{22} or E_{22} plane. Therefore the slowing up of the $S_{22}(j\omega)$ clockwise rotation is equivalent to keeping $S_{22}(j\omega)$ in the neighborhood of its point of tangency over a greater frequency range than $E_{22}(j\omega)$. A detailed discussion, derivation, and interpretation of these statements is given in Reference 7.

References 2, 3 and 7 also show that these requirements on S_{22} are sufficient to insure that D_{22} corresponds to a realizable impedance function, i.e., that D_{22} is the reflection factor of a realizable driving point impedance. This means that $D_{22}(p)$ has no poles in the right-hand half of the p -plane.

Equations (10) and (11) yield:

$$D_{12} = \frac{S_{12}}{E_{12}} \quad (1 - D_{22} E_{11}) \quad (16)$$

$$D_{11} = S_{11} - \frac{D_{12}^2 E_{11}}{1 - D_{22} E_{11}} = S_{11} - \frac{S_{12}^2}{E_{12}^2} E_{11} (1 - D_{22} E_{11}) \quad (17)$$

From equation (16) it may be seen that in order for D_{12} to have no poles in the right-hand half-plane, S_{12} must contain all right-hand zeros of E_{12} with at least the same multiplicity because the factor $(1 - D_{22} E_{11})$ cannot have right-hand zeros. This is due to the maximum modulus Theorem

which states that since $D_{22}(p)$ and $E_{11}(p)$ are known to be analytic in the right-hand half-plane and on the $p = j\omega$ boundary, the maximum value of $|D_{22}(p)E_{11}(p)|$ in the interior of the right-hand plane is less than the maximum value of $|D_{22}(j\omega)E_{11}(j\omega)|$. This latter quantity is known to be no greater than unity because D_{22} and E_{11} correspond to realizable impedances.

From equation (17) it may be seen that D_{11} will not have poles in the right-hand half-plane as long as D_{12} does not have any poles in this region.

Thus, the necessary and sufficient conditions have been established which insure that 2-port D is realizable. These may be summarized in the following Theorem.

Theorem 3

The necessary and sufficient conditions that $[S]$ represent the matrix of the 2-port composed of matching 2-port D in tandem with a given lossless 2-port E are:

- (a) Matrix $[S]$ should be realizable.
- (b) Right-hand and boundary zeros of E_{12} should appear in S_{12} with at least the same multiplicity.
- (c) At each n^{th} order zero of E_{12} , in the interior of the right half plane $(S_{22}-E_{22})$ should have a zero of order $2n$ at least.
- (d) At each boundary zero of E_{12} , $(S_{22}-E_{22})$ has a zero at least of order $2n$, or order $(2n-1)$. In the latter case, the first term in the expansion for $S_{22}-E_{22}$ makes $S_{22}(j\omega)$ have a slower clockwise rotation than $E_{22}(j\omega)$ in the neighborhood of the E_{12} zero.

The restriction on the coefficient of this $(2n-1)$ power in the S_{22} series specified in Part (d) of Theorem 3 is easily interpreted in terms of physical elements in the lossless network E. Any lossless 2-port can be constructed as a chain of simple networks as described by Darlington (Reference 6). These simple network responsible for any boundary zero at the end adjacent to the matching network. When arranged in this way, the coefficient of the $(2n-1)$ power can be identified with either a series reactance pole or a shunt susceptance pole whose effective residue may only be increased by incorporating a similar reactance or susceptance pole in the matching network.

5. Integral Representation of S_{22} Restrictions

It will be recalled that, when the matching problem was stated, $|S_{11}(j\omega)|$ and $|S_{12}(j\omega)|$ were to be quantities of engineering importance. Theorem 3 must be expressed in such a way that it is interpretable in terms of these quantities. Ability to transform parts (c) and (d) of Theorem 3 into integral equations involving $|S_{22}(j\omega)|$ is due to Bode (Reference 1) and Fano (Reference 2,3). All of the Fano work is applicable here because the restrictions on $|S_{22}(j\omega)|$ are independent of whether the matching network D is lossy or lossless. Derivation and tabulation of all the integral formulas involves too much duplication to be repeated here so the reader is referred either to Reference (2) or (3). Formulas used in the illustrative examples of this report will be derived as needed.

All the integral formulas are of the form:

$$\int_0^{\infty} f(\omega) \log \left| \frac{1}{S_{22}} \right| d\omega = F \quad (18)$$

where $f(\omega) = \Phi(j\omega)$ and F involves one of the first $(2n-1)$ coefficients in the expansion for $1/S_{22}(p)$ about a particular right-hand or boundary zero of E_{12} . F also involves right-hand zeros of $S_{22}(p)$, some of which are prescribed, and some of which are inserted as adjustable design parameters. The prescribed zeros in S_{22} are the right-hand zeros of E_{22} which are coincident with E_{12} zeros (See Equation (11)). The function $f(\omega)$ involves the location of a particular E_{12} zero and the order of the coefficient involved in F.

The integral equation (18) may be interpreted as a restriction on the area under the $\ln|1/S_{22}|$ curve plotted with the weighting function $f(\omega)$. An n^{th} order zero of E_{12} at $p = 0$ or $p = \infty$ contributes n such equations, a pair of finite boundary zeros contributes $2n$ equations, a right-hand zero on the real axis contributes $(2n - n_0)$ equations, and a conjugate pair of right-hand zeros contributes $(4n - 2n_0)$ equations where n_0 is the order of a coincident zero in E_{22} .

All of these integral equations must be satisfied simultaneously. This is done by suitable choice of $|S_{22}(j\omega)|$ function, by inserting some arbitrary right-hand zeros in S_{22} , and by changing those F values which correspond to the last controlled coefficients at each boundary E_{12} zero. In accordance with the discussion following Theorem 3, those values of F which can be changed are associated with reactance or susceptance poles whose residues can be increased by elements in the matching network. Satisfaction of all the integral formulas with all prescribed right-hand zeros in S_{22} is equivalent to satisfying parts (c) and (d) of Theorem 3.

6. Magnitude of $[S(j\omega)]$ Elements

It is now possible to derive a theorem concerning the magnitudes of the elements of $[S]$ along the $j\omega$ axis of the p -plane. First, any $|S_{22}(j\omega)|$ function which satisfies the integral restrictions is permitted in a given problem provided $|S_{22}(j\omega)| \leq 1$. From the magnitude function, $S_{22}(p)$ is uniquely determined because all its poles are located in the interior of the left-hand plane, and no interior right-hand zeros are permitted except those specifically inserted to satisfy the integral formulas.

Second, any $|S_{12}(j\omega)|$ function is permitted as long as:

$$1 - |S_{12}(j\omega)|^2 - |S_{22}(j\omega)|^2 \geq 0 \quad (13)$$

because any required right-hand zeros may be inserted in the $S_{12}(p)$ function without changing the magnitude along the $j\omega$ axis. Boundary zeros of E_{12} will automatically appear in S_{12} by virtue of the integral equations and equation (13).

The third consideration, and all that remains, is to find an $|S_{11}(j\omega)|$ function that satisfies Part (a) of Theorem 3, namely that matrix $[S]$ correspond to a realizable 2-port.

In addition to being rational functions of p which are real when p is real, the elements of $[S]$ must have no poles in the right-hand half-plane, and they must satisfy inequalities (13) and (14). Inequality (13) is assumed to be satisfied in the choice of $|S_{12}(j\omega)|$ and $|S_{22}(j\omega)|$ functions. Inequality (14) is rewritten here in slightly different form as:

$$(1 - |S_{12}|^2)^2 - |S_{22}|^2 - |S_{11}|^2 + |S_{11}S_{22}|^2 - 2 \operatorname{Real}(S_{12}^* S_{11} S_{22}) \geq 0$$

$$p = j\omega \quad (19)$$

This inequality involves phases as well as magnitudes because of the last term in the expression. This term can vary between the limits $\pm 2|S_{12}^* S_{11} S_{22}|$ depending on the phase angle of the product. The upper limit is the most favorable because it makes the left-hand side of the inequality as large as possible for a given set of magnitude functions. This is desirable since it permits as large a value of $|S_{12}(j\omega)|^2$ (the measure of power transfer) as possible to be chosen consistent with equation (13). To maintain this favorable upper limit it is necessary that:

$$S_{12}^* S_{11} S_{22} = - |S_{12}^* S_{11} S_{22}| \quad (p = j\omega) \quad (20)$$

Equation (20) imposes conditions only on the phase angles of the three functions because the magnitudes of both sides are identically the same. Solving for S_{11} in terms of its magnitude and the other quantities,

$$S_{11} = -\frac{|S_{12}|^2 |S_{22} S_{11}|}{S_{12}^* S_{22}} \quad (p = j\omega) \quad (21)$$

it can be seen that difficulty in satisfying equation (21) can arise from two sources: first, $|S_{11}(j\omega) S_{22}(j\omega)|$ may not be a rational fraction, (since it involves a square root operation) and second, the S_{11} function specified by equation (21) may have right-hand poles. The right-hand poles can arise from the denominators of $|S_{12}(j\omega)|^2$, $|S_{11}(j\omega)|$, and $|S_{22}(j\omega)|$ or from the numerators of $S_{12}^*(j\omega)$ or $S_{22}(j\omega)$. All of these factors which introduce right-hand poles in $S_{11}(p)$ can be cancelled out by multiplying $S_{12}(p)$ by unit magnitude phase-shift factors of the form:

$$\frac{p - p_x}{p + p_x^*} \quad (22)$$

where the various p_x are undesired roots located in the interior of the right-hand half-plane.

The first difficulty mentioned above is more serious: $|S_{11}(j\omega) S_{22}(j\omega)|$ may not be a rational fraction because $|S_{11}(j\omega) S_{22}(j\omega)|^2$ may not be a perfect square even though this latter quantity is rational. There are many ways in which irrationality is avoided. For example, if S_{11} is identically zero, the last three terms drop out of inequality (19) which is then easily simplified. In the case of a lossless network $|S_{11}(j\omega)| = |S_{22}(j\omega)|$ so that the product is rational.

In the general case where the simplest rational function whose amplitudes give the desired $|S_{11}(j\omega)|$ and $|S_{22}(j\omega)|$ do not have a rational fraction product it is necessary to approximate one or both of these functions with (usually) more complicated algebraic functions such that their product is rational. This requirement means that the product $S_{11}(p) S_{22}(p)$ must be factorable into perfect squares and pairs of factors symmetrical about the $j\omega$ axis. The symmetrical pairs may appear as a product in either the numerator or denominator, or they may appear as a quotient, i.e. one factor in the numerator and one in the denominator. The prescribed right-hand zeros of S_{22} must, of course, appear since these are prescribed by the integral formulas.

This requirement on the product of $|S_{11}|$ and $|S_{22}|$ makes it more difficult, but not impossible to fit rational fractions to the desired $|S_{11}(j\omega)|$ and $|S_{22}(j\omega)|$ functions for the optimum case where equation (21) is to be satisfied. As long as this approximation procedure can be carried out with an arbitrarily small error, the inequality may be written in its most favorable form, which contains only magnitudes.

$$(1 - |S_{12}|^2) - |S_{22}|^2 - |S_{11}|^2 + |S_{11}S_{22}|^2 + 2|S_{12}^2S_{11}S_{22}| \geq 0 \quad (p = j\omega) \quad (23)$$

This expression can be factored into

$$\left\{ |S_{12}|^2 - (1 + |S_{11}|)(1 - |S_{22}|) \right\} \left\{ |S_{12}|^2 - (1 - |S_{11}|)(1 + |S_{22}|) \right\} \geq 0 \quad (p = j\omega) \quad (24)$$

Inequality (24) could be rearranged to give $|S_{11}(j\omega)|$ explicitly in terms of the other two quantities, but it is easier to leave it implicitly specified. That is, assume that $|S_{11}(j\omega)|$ and $|S_{22}(j\omega)|$ are specified and that both are no greater than unity and, in addition, $|S_{22}(j\omega)|$ satisfies all integral formulas. The expression in (24) is quadratic in $|S_{12}(j\omega)|^2$ and there are always two positive, real roots which coincide when $|S_{11}(j\omega)| = |S_{22}(j\omega)|$. Inequality (24) can be satisfied by making both factors negative or both factors positive. In order for both factors to be positive, however, $|S_{12}(j\omega)|^2$ would have to be positive, however, $|S_{12}(j\omega)|^2$ would have to be so large that it would violate inequality (13). Therefore, both factors must be negative, which means that $|S_{12}(j\omega)|^2$ must be equal to or less than the smaller of the two roots as follows:

$$\text{If } |S_{11}(j\omega)| \leq |S_{22}(j\omega)|$$

$$|S_{12}(j\omega)|^2 \leq (1 + |S_{11}(j\omega)|)(1 - |S_{22}(j\omega)|) \quad (25)$$

$$\text{If } |S_{11}(j\omega)| > |S_{22}(j\omega)|$$

$$|S_{12}(j\omega)|^2 < (1 - |S_{11}(j\omega)|)(1 + |S_{22}(j\omega)|) \quad (26)$$

Satisfaction of the appropriate inequality, (25) or (26), for any frequency range automatically insures satisfaction of inequality (13), so that only inequalities (25) and (26) need to be considered in the following Theorem.

Theorem 4

The input reflection coefficient magnitude $|s_{11}(j\omega)|$ and the transmission coefficient $|s_{12}(j\omega)|^2$ can always be approximated with a structure consisting of a physically realizable matching network in tandem with a prescribed lossless 2-port \mathbb{E} if:

$$(a) (1) |s_{12}(j\omega)|^2 \leq (1 + |s_{11}(j\omega)|)(1 - |s_{22}(j\omega)|)$$

$$\text{if } |s_{11}(j\omega)| \leq |s_{22}(j\omega)|$$

$$(2) |s_{12}(j\omega)|^2 \leq (1 - |s_{11}(j\omega)|)(1 + |s_{22}(j\omega)|)$$

$$\text{if } |s_{11}(j\omega)| > |s_{22}(j\omega)|$$

(b) $|s_{22}(j\omega)|$ satisfies all the integral restrictions imposed by the lossless 2-port \mathbb{E} with all necessary right-hand zeros inserted in the $S_{22}(p)$ function.

(c) The magnitude functions can be obtained exactly if they are specified as rational fractions in ω and $|s_{11}(j\omega)s_{22}(j\omega)|$ is a rational fraction in ω .

Thus the limitations on the performance of a matching network are given in a simple form which can be easily interpreted in terms of quantities of direct engineering importance.

7. Optimum Matching Network

In a particular matching problem there is some idea or specification of the maximum allowable input reflection coefficient magnitude $|s_{11}(j\omega)|$ over a certain range of frequencies. Outside this frequency range the input mismatch may be of no interest. There are, of course, many applications where it is desirable that the mismatch be small over the infinite frequency spectrum. The case where the input mismatch is zero over the infinite frequency band is discussed in Reference 10. There is usually a particular shape of $|s_{12}(j\omega)|$ curve appropriate to the given problem. It is desired to know what amplitude scale is permitted for $|s_{12}(j\omega)|$, that is, knowing the general shape of $|s_{12}(j\omega)|$, determine how great $|s_{12}(j\omega)|$ can be made. Sometimes,

the desired performance is easily obtained and the maximum value of $|S_{12}(j\omega)|$ can be made unity. In such cases the performance satisfies Theorem 4, but does not reach the ultimate limit set by the theorem, in the sense that the resulting input mismatch may be larger than the minimum permitted by the theorem.

This section applies only to matching networks whose performance is limited by Theorem 4. The network giving the largest $|S_{12}(j\omega)|$ curve when limited by Theorem 4 is called the "optimum matching network from the standpoint of power transmission efficiency".

In order to predict the form of the optimum matching network it is necessary to repeat here the order in which restriction are imposed on the three magnitude functions, $|S_{22}(j\omega)|$, $|S_{11}(j\omega)|$, and $|S_{12}(j\omega)|$. First, $|S_{22}(j\omega)|$ is restricted by the given load through the integral equations. These have the effect of setting a minimum function for $|S_{22}(j\omega)|$. The function may be reduced in any small range of ω , but it will have to be increased at some other frequency in order to satisfy the integral restrictions.

Second, presuming $|S_{22}(j\omega)|$ has been chosen as above, $|S_{11}(j\omega)|$ should always be equal to or less than $|S_{22}(j\omega)|$. For, if inequality (a) (2) of Theorem 4 is applicable, it may be transformed into

$$|S_{12}(j\omega)|^2 < 1 - |S_{11}(j\omega)|^2 < 1 - |S_{22}(j\omega)|^2 \quad (27)$$

which shows that the $|S_{12}(j\omega)|$ permitted is smaller than would be allowed if $|S_{11}(j\omega)|$ and $|S_{22}(j\omega)|$ were equal. Over portions of the frequency band where $|S_{11}(j\omega)|$ can be made large, $|S_{22}(j\omega)|$ should be made at least as large in order to conserve area under the $|1/S_{22}(j\omega)|$ curve. This permits decreasing $|S_{22}(j\omega)|$ at some other frequency where a larger permissible value of $|S_{12}(j\omega)|$ is desired.

Thus, the optimum matching network from the standpoint of power transmission causes inequality (a)(1) of Theorem 4 to be applicable at all frequencies.

The third characteristic of the optimum matching network is that it satisfies at all frequencies the equation:

$$|S_{12}(j\omega)|^2 = (1 + |S_{11}(j\omega)|) (1 - |S_{22}(j\omega)|) \quad (28)$$

which is the limiting case of the inequality (a)(1) of Theorem 4. For, if a matching network is obtained which satisfies the inequality but not the equality, the power transmission can immediately be improved by multiplying

$|S_{12}(j\omega)|$ by a constant large enough to cause equation (28) to be satisfied at one or more discrete frequencies. At these frequencies $|S_{11}(j\omega)|$ can be increased to allow further amplification of the $|S_{12}(j\omega)|$ curve. If $|S_{11}(j\omega)|$ is already at its prescribed limit, $|S_{22}(j\omega)|$ can be decreased with a compensating increase at some other frequency and/or relocation of the non-prescribed $S_{22}(p)$ right-hand zeros to keep the integral equations satisfied. The effect of such an adjustment is to decrease the right-hand side of equation 1(a) of Theorem 4 at frequencies where the inequality applies, and to increase the right-hand side where the equality applies.

For any $|S_{11}(j\omega)|$ and $|S_{22}(j\omega)|$ function, equation (28) specifies an upper limit on the $|S_{12}(j\omega)|$ function. At any step in the improvement process described above, the actual $|S_{12}(j\omega)|^2$ curve lies below this upper limit except at points of tangency. Each change brings the two curves closer together: the actual $|S_{12}(j\omega)|$ curve increases without changing shape, while the maximum allowable $|S_{12}(j\omega)|^2$ defined by equation (28) changes shape as it conforms more closely to the actual $|S_{12}(j\omega)|^2$ curve. Eventually the two curves coincide and the matching network satisfies equation (28) identically. Further improvement is not possible, so the matching network which satisfies equation (28) is the optimum matching network from the standpoint of power transmission efficiency. Theorem 5 can now be stated.

Theorem 5

The optimum matching network from the standpoint of power-transmission efficiency has the following performance.

$$(a) |S_{11}(j\omega)| \leq |S_{22}(j\omega)|$$

$$(b) |S_{12}(j\omega)|^2 = (1 + |S_{11}(j\omega)|)(1 - |S_{22}(j\omega)|)$$

(c) $|S_{22}(j\omega)|$ satisfies all integral restrictions imposed by the given load.

This performance can always be approximated with an arbitrarily small error by a physically realizable matching network. When $|S_{11}(j\omega)S_{22}(j\omega)|$ is a rational fraction in ω , the performance can be exactly obtained.

The optimum matching network, although not unique, has a distinctive form because satisfaction of equation (28) means that the matrix $[1 - S^*(j\omega)S(j\omega)]$ is singular for all ω . This means that the open-circuit resistance matrix is singular if the overall network S has an impedance matrix. If the short-circuit admittance matrix exists, satisfaction of equation (28) means that the short-circuit conductance matrix is singular.

Except for the case of an ideal transformer, every 2-port must have either an open-circuit impedance matrix or a short-circuit admittance matrix. The 2-port S of the matching problem is the tandem combination of the matching 2-port D and the lossless 2-port E which represents the load impedance $Z(p)$. The lossless 2-port E is always more complicated than an ideal transformer; hence the tandem combination must possess either an impedance or admittance matrix.

The matching network D will also have a singular open-circuit resistance or short-circuit conductance matrix, depending on whether the impedance or admittance matrix, or both, exist. In the general Gewertz process (Reference 9) boundary poles of impedance or admittance are removed and the matrix is inverted. The singular real part matrix means that at each inversion the new admittance or impedance matrix will have boundary poles in all its elements. The successive inversion and removal of boundary poles is continued until the remainder of the network to be synthesized consists only of a single 1-port in series or in shunt with one winding of an ideal transformer. The 1-port is completely specified by its terminal impedance and may be synthesized in a number of forms including that of a lossless 2-port terminated in a single resistor. If the matching network is lossless, the single resistor can be considered to be present but not connected to the main part of the network. This can be summarized in a theorem.

Theorem 6

Any "optimum matching network from the standpoint of power transmission efficiency" may be constructed with no more than one resistor.

A reasonable speculation which has not been rigorously proved is that if an equivalent for a one resistor optimum matching network is constructed using more than one resistor then all resistances belong to a single 1-port. It has not been possible to find any 2-port having a singular real-part matrix which violates the statement. Furthermore, all known methods of 2-port synthesis lead to this form of network.

The foregoing theory will now be illustrated by means of two examples.

Example 1

One of the simplest examples of a load impedance consists of a resistor and inductor in series as shown in Fig. MRI-13040a. The resistor is assumed to be one ohm in this example. It will also be assumed that the useful frequency range is from $\omega = 0$ to $\omega = 1$ and that a matching network is to be designed such that over this range, a one-ohm generator having 1 watt available power sees a small mismatch.

Suppose that the magnitude of the reflection coefficient $S_{11}(j\omega)$ seen by the generator is to be held constant over the infinite frequency band at some value $|S_{11}|$ as shown in Fig. MRI-13040c. Suppose further that the power in the load $P(\omega)$ is to be constant and equal to P_0 for $0 \leq \omega \leq 1$ and can be zero for $\omega > 1$ as shown in Fig. MRI-13040b. The maximum value of $P(\omega)$ can now be computed without regard to the complexity of the matching network.

From Theorem 5, it is known that the optimum condition from the standpoint of power transmission is to have:

$$|S_{22}(j\omega)| \geq |S_{11}| \quad (29)$$

$$P(\omega) = |S_{12}(j\omega)|^2 = (1 + |S_{11}|)(1 - |S_{22}(j\omega)|) \quad (30)$$

For the frequency range $0 \leq \omega \leq 1$, the highest value of P_0 is obtained by making $|S_{22}(j\omega)|$ constant and as small as possible. For $\omega > 1$, $|S_{22}(j\omega)|$ can be unity so that $\log |1/S_{22}(j\omega)|$ is zero outside the pass-band. This will be seen to make the most efficient use of the integral restriction on $|S_{22}(j\omega)|$. This integral restriction will now be derived.

It will be noted that the load in this example is so simple that the lossless 2-port E required for the Darlington representation consists simply of the series inductance L . The elements of the scattering matrix $|E(p)|$ are given by

$$E_{11}(p) = E_{22}(p) = \frac{Z(p) - 1}{Z(p) + 1} = \frac{pL}{pL + 2} \quad (31)$$

$$E_{12}(p) = \frac{2}{pL + 2} \quad (32)$$

$E_{12}(p)$ has a simple zero at infinity. The expansion for E_{22} in terms of $1/p$ is

$$E_{22} = 1 - \frac{2}{pL} + \dots \quad (33)$$

From Theorem 3, $S_{12}(p)$ must have at least a simple zero at infinity and also since a boundary zero is concerned here $(S_{22} - E_{22})$ must have at least a simple zero at infinity. This means that the $1/p$ series representing $(S_{22} - E_{22})$ should start with a $1/p$ term unless the $1/p$ term

in the S_{22} series is the same as the corresponding term in the E_{22} series.

That is if the series for S_{22} is

$$S_{22} = 1 - \frac{2}{pL_1} + \dots \quad (34)$$

L_1 may be equal to L , or L_1 may be larger than L . The latter condition causes the $S_{22}(p)$ vector to have a slower clockwise rotation than $E_{22}(p)$ as p approaches infinity. The simple physical interpretation for this is that the matching network can add an inductance ($L_1 - L$) in series with L so that when looking into the number 2 port of the composite network S , an inductance greater than or equal to L may be seen. In this simple problem it would be senseless to add inductance in series with the element that is to be matched out so for this example L_1 will be assumed equal to L and the series for S_{22} will be taken as:

$$S_{22} = 1 - \frac{2}{pL} + \dots \quad (35)$$

If the line integral

$$\int \log \left| \frac{1}{S_{22}} \frac{\pi}{2} \frac{p - p_x}{p + p_x} \right| dp \quad (36)$$

is taken around the clockwise path shown in Fig. MRI-13043a the result will be zero provided $(p - p_x)$ factors are inserted as shown to cancel all right-hand zeros of $S_{22}(p)$. The expansion for the integrand is known on the large semi-circle of radius R as:

$$\log \left[\frac{1}{S_{22}} \frac{\pi}{2} \frac{p - p_x}{p + p_x} \right] = \log \left[1 + \left(\frac{2}{L} - 2 \frac{p}{p_x} \right) \frac{1}{p} + \dots \right] = \left(\frac{2}{L} - 2 \frac{p}{p_x} \right) \frac{1}{p} + \dots \quad (37)$$

The integral along the indented path on the $p = j\omega$ axis is equal to the integral on the large semi-circle taken counter-clockwise which is

$$2\pi j \left(\frac{1}{L} - \frac{2}{p_x} \right) \quad (38)$$

The small indentations on the $p = j\omega$ path are to keep boundary singularities of the integrand out of the contour. These are actually not necessary because the singularities are only logarithmic and are integrable. The real part of the integrand is even while the imaginary part is odd.

The oddness of the imaginary part is assured by the fact that :

$$\frac{1}{S_{22}} \propto \frac{p - p_x}{p + p_x}$$

contains no right-hand factors in either numerator or denominator. Therefore the sign of this quantity is positive at $p = 0$ and there is no $j\omega$ term associated with the logarithm of this quantity. Such a $j\omega$ term would destroy the odd symmetry of the imaginary part.

The integral of the odd part cancels out over the whole path along the $p = j\omega$ axis so that only the even part is left giving

$$\int_{-\infty}^{\infty} \log \left| \frac{1}{S_{22}} \right| j d\omega = 2\pi j \left(\frac{1}{L} - \frac{1}{X} p_x \right) \quad (39)$$

which simplifies to the desired integral restriction.

$$\int_0^{\infty} \log \left| \frac{1}{S_{22}} \right| d\omega = \pi \left(\frac{1}{L} - \frac{1}{X} p_x \right) \quad (40)$$

It may be seen from (40) that any right-hand zeros in S_{22} will require p_x terms which will have the effect of decreasing the integral just as if the value of L were increased. Therefore $S_{22}(p)$ should contain only those right-hand zeros which are prescribed and in this case there are none.

At the outset, it was stated that $\log |1/S_{22}|$ would be constant for $0 < \omega \leq 1$ and would be zero for $\omega > 1$. This makes the integral very simple to evaluate, giving the minimum value for $|S_{22}(j\omega)|$ in the range $0 < \omega \leq 1$. This value is

$$|S_{22}(j\omega)| = e^{-\pi/L} \quad 0 < \omega \leq 1 \quad (41)$$

and therefore the maximum value of P_o is known in terms of the specified mismatch $|S_{11}|$ and specified load inductance as

$$P_o = (1 + |S_{11}|)(1 - e^{-\pi/L}) \quad |S_{11}| \leq e^{-\pi/L} \quad (42)$$

It can be seen from (42) that a slight increase in P_o can be obtained by increasing $|S_{11}|$, but for large L , the allowable mismatch is restricted to a very small value, $e^{-\pi/L}$. It will be recalled that making $|S_{11}|$ greater than this value would cause the wrong inequality of Theorem 4 to be

applicable and the matching network would not be the optimum defined by Theorem 5.

The $P(\omega)$, $|S_{11}(j\omega)|$ and $|S_{22}(j\omega)|$ curves are given by Figures MRI-13040b, MRI-13040c and MRI-13040d respectively. A matching network would have to be quite complicated to produce such sharp transitions as are shown in these curves. To illustrate the procedure of actually designing a network, a simple approximating function, will be used for $P(\omega)$ and the matching network which produces this function will be shown.

The function which will be used to approximate the rectangular curve of Fig. MRI-13040b is:

$$|S_{12}(j\omega)|^2 = P(\omega) = \frac{a^2}{1 + \omega^2} \quad (43)$$

which has the value a^2 at $\omega = 0$ and drops to half this value at $\omega = 1$, the edge of the pass band. This is not a very close approximation to the desired rectangular form, but the algebraic complexity of better approximation makes them too formidable.

The input mismatch will again be taken as constant, that is

$$|S_{11}(j\omega)| = |S_{11}|.$$

From equation (30) the $|S_{22}(j\omega)|$ function may be determined as :

$$|S_{22}(j\omega)| = \frac{1 - \frac{a^2}{1 + |S_{11}|} + \omega^2}{1 + \omega^2} \quad (44)$$

The complex function $S_{22}(p)$ is easily determined from the knowledge that $S_{22}(p)$ should have no right-hand zeros in order to make the most efficient use of the integral restriction.

$$S_{22}(p) = \frac{\left(p + \sqrt{1 - \frac{a^2}{1 + |S_{11}|}} \right)^2}{(p + 1)^2} \quad (45)$$

The simplest possible $S_{12}(p)$ is

$$S_{12}(p) = \frac{a}{p+1} \quad (46)$$

Now $S_{11}(p)$ is specified according to equation (21) as:

$$S_{11}(p) = \frac{|S_{11}| \left(p - \sqrt{1 - \frac{a^2}{1 + |S_{11}|}} \right)}{p + \sqrt{1 - \frac{a^2}{1 + |S_{11}|}}} \quad (47)$$

and, fortunately, this turns out to be rational.

The $[S]$ matrix, whose elements are specified above, is the matrix of the 2-port composed of the matching network and the inductance L of the load. If the 2-port E representing the load were complicated, the easiest method of synthesis would be to find the scattering or impedance matrix of the matching network D . The matching network would then be synthesized by itself. In this case, however, the simplest procedure is to find the impedance matrix of the composite network $[S]$ and then synthesize the composite network. The impedance matrix is used because it is known at the outset that there will be an inductance L in series with one terminal of the network.

Matrix equation (5) gives the impedance matrix in terms of the scattering matrix. The matching network and the element values obtained from this impedance matrix are shown in Fig. MRI-1304la.

It should be noted that the relation between a , $|S_{11}|$ and L could be found from the integral restriction on $|S_{22}(j\omega)|$. This would involve complicated mathematics, however, so it is better to proceed with the synthesis and obtain a value for L in terms of a and $|S_{11}|$. The value in this case is

$$L = \frac{1}{1 - \sqrt{1 - \frac{a^2}{1 + |S_{11}|}}} \quad (48)$$

This gives the value:

$$a^2 = (1 + |S_{11}|) \left(\frac{2}{L} - \frac{1}{L^2} \right) \quad (49)$$

which may be compared with the height of the rectangular curve given by Equation (42).

It was noted in this case, the minimum value of $|S_{22}(j\omega)|$ occurs at $\omega = 0$ and this value must be greater than $|S_{11}|$. That is

$$1 - \frac{a^2}{1 + |S_{11}|} \geq |S_{11}| \quad (50)$$

which can be changed to

$$1 - |S_{11}|^2 - a^2 \geq 0 \quad (51)$$

Using the value for a^2 given by Equation (49)

$$(1 + |S_{11}|)(1 - |S_{11}| - \frac{2}{L} + \frac{1}{L^2}) \geq 0 \quad (52)$$

Equation (52) may be interpreted in two ways: as an upper limit on $|S_{11}|$, and as a lower limit on L . When the inequality is solved for L_1 the relevant branch of the solution is

$$L \geq \frac{1}{1 - \sqrt{|S_{11}|}} \quad (53)$$

For large values of L , the performance of the simple network may be compared with the rectangular curve of Fig. MRI-1304Ob because the exponential in equation (42) may be replaced by a series to give:

$$P_o \approx (1 + |S_{11}|) \frac{\pi}{L} \quad L \gg 1 \quad (54)$$

while equation (49) may be approximated by:

$$a^2 \approx (1 + |S_{11}|) \frac{2}{L} \quad L \gg 1 \quad (55)$$

It may be seen that for large L_1 the $P(\omega)$ of the simple network is only $2/\pi$ of the rectangular response at $\omega = 0$ and at $\omega = 1$ the ratio is worse, namely $1/\pi$. For small values of L_1 the simple circuit of Fig. MRI-1304la compares a little better. The response is shown in Fig. MRI-1304lb for the smallest possible value of L , namely $L = 1$ which requires $|S_{11}| = 0$ and $a^2 = 1$.

These conditions show the simple network of Fig. MRI-13041a in its most favorable light for comparison with the rectangular response.

The choice of parameters used in the above plot leads to a rather simple degenerate case. The matching network consists of the series combination of a resistor and capacitor in parallel with the load as indicated in Fig. MRI-13041b and is a well known R-L-C constant resistance network. The ideal transformers both disappear only for this combination of parameters. The lower transformer of Fig. MRI-13041a can be eliminated if $a = 1$ and $|S_{11}| = 0$, while only the upper transformer disappears for $aL = 1 + |S_{11}|$. For both transformers to disappear, it is necessary that $a = 1$, $|S_{11}| = 0$, and $L = 1$.

Example 2

Suppose that a matching network is to be designed to match the load impedance of Fig. MRI-13042a to a one ohm generator so that the generator sees a perfect match at all frequencies and so that the power reaching the load is constant and independent of frequency. That is $P(\omega) = P_0$ for all ω .

There are two answers that might be desired in this problem. The first is a knowledge of the maximum value of P_0 without any knowledge of the form of the matching network. The second answer is a complete design of the optimum matching network.

The first answer is obtained by means of the integral restrictions on the magnitude of $S_{22}(j\omega)$.

The lossless 2-port which, when terminated in one ohm, represents $Z(p)$ may be obtained as follows:

$$Z(p) = \frac{2p + 1}{p + 1} \quad (56)$$

$$E_{11}(p) = \frac{Z(p) - 1}{Z(p) + 1} = \frac{p}{3p + 2} \quad (57)$$

$$|E_{12}(j\omega)|^2 = 1 - |E_{11}(j\omega)|^2 = 1 - \frac{\omega^2}{9\omega^2 + 4} = \frac{8\omega^2 + 4}{9\omega^2 + 4} \quad (58)$$

$$E_{12}(p) = \frac{-2(\sqrt{2}p - 1)}{3p + 2} \quad (59)$$

The above $E_{12}(p)$ function is not the only one which has the magnitude specified by (58) but it is the simplest function. The function $E_{22}(j\omega)$ is now obtained from the requirement that the scattering matrix $[E]$ of the lossless 2-port be unitary on the $p = j\omega$ axis. This same requirement was used to obtain $|E_{12}(j\omega)|$ according to equation (58).

$$E_{22}(j\omega) = -\frac{E_{11}^*(j\omega)E_{12}(j\omega)}{E_{12}^*(j\omega)} = -\frac{-j\omega(\sqrt{Z}j\omega-1)(2-3j\omega)}{(2-3j\omega)(3j\omega+2)(-\sqrt{Z}j\omega-1)} \quad (60)$$

$$E_{22}(p) = \frac{-p(\sqrt{Z}p-1)}{(3p+2)(\sqrt{Z}p+1)} \quad (61)$$

The actual lossless 2-port E may be found from the above scattering matrix by determining the open circuit impedance matrix from equation (5) and then synthesizing the network from the impedance description. The 2-port E is shown in Fig. MRI-13042b.

The function $E_{12}(p)$ has a simple zero on the positive real axis and the reflection coefficient $E_{22}(p)$ has a coincident zero. That is, $n = 1$ and $n_0 = 1$. Therefore, there should be $2n - n_0 = 1$ integral restriction on $|S_{22}(j\omega)|$. This restriction will now be derived.

Theorem 3 requires that $S_{22} - E_{22}$ have at least a second order zero at $p = 1/\sqrt{Z}$. Such a second order zero is assured by putting a zero in $S_{22}(p)$ at $p = 1/\sqrt{Z}$ and making the constant term (first term) in the Taylor series for $E_{22}(p)/\sqrt{Z}p-1$ equal to the first term in the Taylor series for $S_{22}(p)/\sqrt{Z}p-1$.

The following line integral is taken around the path of Fig. MRI-13043a.

$$\int \frac{\log \left[\frac{1}{S_{22}} \left(\frac{\sqrt{Z}p-1}{\sqrt{Z}p+1} \right) \pi \frac{p-1}{p+\sqrt{Z}p} \right]}{\left(p + \frac{1}{\sqrt{Z}} \right) \left(p - \frac{1}{\sqrt{Z}} \right)} dp \quad (62)$$

The argument of the logarithm in the integral numerator has the same magnitude as $1/S_{22}$ along $p = j\omega$ but all right-hand zeros have been cancelled out of S_{22} .

If the numerator is expanded in a Taylor series at $p=1/\sqrt{Z}$, the first term in the expansion is known by virtue of the second order zero in $S_{22}-E_{22}$.

The higher order terms contain powers of $(p - 1/\sqrt{2})$ which cancel out the $p - 1/\sqrt{2}$ factor in the denominator. Thus, only the first term has a residue at $p = 1/\sqrt{2}$ and the following equation results:

$$\oint \frac{\log \left[\frac{1}{S_{22}} \frac{\sqrt{2}p-1}{\sqrt{2}p+1} \frac{\pi}{2} \frac{p-p_x}{p+p_x} \right]}{\left(p - \frac{1}{\sqrt{2}} \right) \left(1 + \frac{1}{\sqrt{2}} \right)} dp = \oint \frac{\log \left[-(3+2\sqrt{2}) \frac{\pi}{2} \frac{\frac{1}{\sqrt{2}} - p_x}{\frac{1}{\sqrt{2}} + p_x} \right]}{\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \left(p - \frac{1}{\sqrt{2}} \right)} dp$$

$$= \frac{2\pi j}{\sqrt{2}} \log \left[-(3 + 2\sqrt{2}) \frac{\pi}{2} \frac{\frac{1}{\sqrt{2}} - p_x}{\frac{1}{\sqrt{2}} + p_x} \right] \quad (64)$$

The imaginary part of the logarithm is an odd function along $p = j\omega$ although there may also be a constant $j\omega$ term due to a (-1) factor in $S_{22}(p)$. If such a factor is present in $S_{22}(p)$, however, the right-hand side will also have a $j\omega$ term so these constant terms will cancel out. The integral of the imaginary part on the left-hand side will therefore cancel over the full $j\omega$ axis.

The integral on the left-hand side of (64) is zero on the large semi-circle of the path because the denominator is quadratic. All that remains of the left-hand side of (64), then, is the integral of the real part which simplifies to :

$$\int_0^\infty \frac{\log \left| \frac{1}{S_{22}} \right|}{1 + 2\omega^2} d(\sqrt{2}\omega) = \frac{\pi}{2} \log(3 + 2\sqrt{2}) + \frac{\pi}{2} \oint \log \left| \frac{\frac{1}{\sqrt{2}} - p_x}{\frac{1}{\sqrt{2}} + p_x} \right|$$

$$d(\sqrt{2}\omega) \quad (65)$$

Equation (65) is the desired integral restriction. The p_x in this restriction all have positive real parts so it may be seen that any extra right-hand zeros in $S_{22}(p)$ will decrease the area allowed under the weighted $\log \left| 1/S_{22} \right|$ curve. Therefore, the only right-hand zero which should appear in $S_{22}(p)$ is the required one at $p = 1/\sqrt{2}$.

For the optimum matching network, $|S_{22}(j\omega)|$ should be constant to give a constant $P(\omega)$ according to

$$P(\omega) = P_0 = 1 - |S_{22}(j\omega)| \quad (66)$$

and the value of $|S_{22}(j\omega)|$ is easily obtained because $\log |1/S_{22}|$ can be taken outside of the integral in (65).

$$|S_{22}(j\omega)| = \frac{1}{3+2\sqrt{2}} \quad (67)$$

$$P_o = \frac{2+2\sqrt{2}}{3+2\sqrt{2}} = |S_{12}(j\omega)|^2 \quad (68)$$

The complex function $S_{12}(p)$ is obtained from the known magnitude by noting that $S_{12}(p)$ should contain the $E_{12}(p)$ zero at $p = 1/\sqrt{2}$ with at least the same multiplicity. The simplest function having the proper magnitude along $p = j\omega$ is given in (69). The (-1) factor is inserted to insure that there is no polarity reversal at d.o.

$$S_{12}(p) = -\sqrt{\frac{2+2\sqrt{2}}{3+2\sqrt{2}}} \frac{(\sqrt{2}p - 1)}{\sqrt{2}p + 1} \quad (69)$$

The $S_{22}(p)$ function must also have a zero at $p = 1/\sqrt{2}$. Moreover, the first term in the Taylor series for $S_{22}(p)$ about $p = 1/\sqrt{2}$ must be the same as the corresponding term in the $E_{22}(p)$ series. This fixes the sign of $S_{22}(p)$, giving:

$$S_{22}(p) = \frac{-1}{3+2\sqrt{2}} \frac{(\sqrt{2}p - 1)}{\sqrt{2}p + 1} \quad (70)$$

Now that the overall scattering matrix $[S]$ is completely determined, the scattering matrix $[D]$ of the matching network alone may be found by solving equations (9), (10) and (11) for elements of $[D]$ in terms of $[S]$ and $[E]$. The open-circuit impedance matrix of the matching network can then be computed and the network drawn from the impedance matrix. The matching network is shown in Fig. MRI-13042c and it is seen to contain two ideal transformers. If the minus sign had not been inserted in $S_{12}(p)$, both transformers would have had reversed polarities.

The optimum matching networks in both examples are probably too complicated to use in actual equipment because they contain inter-connected ideal transformers. The chief use of the theory developed in this section is to predict the maximum performance that can be obtained from a matching network,

regardless of its form or complexity. Then the performance of practical matching networks can be compared with this theoretical maximum to determine the amount of improvement that may be afforded by a more ingenuous design. Some practical forms of matching networks are described in Reference 11. Reference 7 illustrates the use of the general theory of wideband matching in guiding the design of simple matching networks.

8. Extension to Arbitrary Generators

The previous sections were concerned with matching an arbitrary load impedance to a generator whose available power was one watt and whose internal impedance was a pure resistance of one ohm. Using a one ohm generator merely means that all impedances are normalized to the internal resistance of the generator. Thus, the theory applies to any problem in which the internal impedance of the given generator is a constant resistance.

The theory already developed can also be extended to include generators whose available power is not constant. These problems occur in practice. For example, consider a generator which consists of a long coaxial cable connected to a constant-voltage, adjustable-frequency oscillator. The cable has a constant, real characteristic impedance because its attenuation per foot is small. However, the cable is long enough so that its output impedance is not affected by the oscillator connected at the input end. Thus the combined oscillator and cable is a generator whose internal impedance is a real constant equal to the characteristic impedance of the cable, but whose available power varies with frequency because the cable attenuation varies with frequency.

The generator in question is shown in Fig. MRI-13043b together with the matching network and the load impedance. The internal impedance of the generator appears as one ohm because all impedances are normalized to the generator resistance. The available power of the generator is

$$P_a(\omega) = \frac{E^2(\omega)}{4} \quad (71)$$

and the power dissipated in the load is $P(\omega)$. $P(\omega)$ and $P_a(\omega)$ are related by:

$$P(\omega) = |S_{12}(j\omega)|^2 P_a(\omega) \quad (72)$$

when $S_{12}(p)$ is the off-diagonal element of the scattering matrix of the tandem combination of matching network D and lossless 2-port E as shown in Fig. MRI-13043c. The 2-port E is used, as before, in the Darlington representation of the load $Z(p)$. Everything is the same as in the previous sections except that $P(\omega)$ is not equal to $|S_{12}(j\omega)|^2$ because the available power of the

generator is no longer one watt. All the Theorems were purposely written in terms of $|S_{12}(j\omega)|^2$ rather than $P(\omega)$ so that they would be applicable to the more general situations.

To summarize: When an arbitrary load impedance $Z(p)$ is to be matched to a one ohm generator whose available power is $P_a(\omega)$, all the Theorems previously developed apply and it is merely necessary to note that $|S_{12}(j\omega)|^2$ is the ratio of power dissipated in the load to available power of the generator.

The theory developed in previous sections applies to the system of Fig. MRI-13044a in which a constant resistance load is to be matched to a generator whose available power and internal impedance are both arbitrary function of frequency.

The available power P_a of the generator in Fig. MRI-13044a is

$$P_a(\omega) = \frac{|E(\omega)|^2}{4R(\omega)} \quad (73)$$

where

$$Z(j\omega) = R(\omega) + jX(\omega) \quad (74)$$

The internal impedance $Z(p)$ of the generator can be represented as a lossless 2-port E terminated in one ohm as shown in Fig. MRI-13044b. Now the system consists of two 2-ports in tandem in exactly the same way as the preceding cases, except for the generator $E(\omega)$ inserted between the two networks.

The phase of the power reaching the load is not important, and since magnitudes only are of interest, the voltage generator can always be placed in series with the terminating resistor at port 2 of network E . This is shown in Fig. MRI-13044c which is identical to Fig. MRI-13043c except that the generator is connected to end 2 of the combined network rather than end 1. This is of no importance because the networks obey reciprocity so that the power $P(\omega)$ in the load is given, as before, by equation (72).

As far as the matching problem is concerned, nothing has been lost by moving the generator from its original position in series with end 1 of the lossless 2-port E to the new position at end 2 of E . Phase information, it is true, has been thrown away, but the magnitudes have been unaltered. The change from Fig. MRI-13044b to Fig. MRI-13044c is always valid as long as $P_a(\omega)$ is finite. This precludes the case in which the internal resistance $R(\omega)$ of the given generator is zero while the voltage $E(\omega)$ is not zero. Such a generator is not physically possible, so that the representation of Fig. MRI-13044c

is always valid.

In summary, then, the general theory of wideband matching applies to the matching of a constant resistance load to a generator whose internal impedance is complex and whose available power is a function of frequency. The mismatch at the terminals of the given constant resistance is given by $|S_{11}(j\omega)|$.

9. Special Case of Reflectionless Matching

Some mention should be made of the important special case of reflectionless matching with optimum power transfer efficiency networks. Such structures match a load to a constant resistance over the infinite real frequency band.

The limitations for this case are readily found from Theorem 5, by introducing the matching constraint $|S_{11}| = 0$, corresponding to a unit normalized constant input resistance. Equation (b) of the Theorem specifying maximum insertion gain becomes

$$|S_{12}(j\omega)|^2 = 1 - |S_{22}(j\omega)| \quad (75)$$

Further the synthesis of the matching network is considerably simplified in this case for the specification of S_{11} as a function of two amplitudes (See Equation 21) whose product must be rational is not required. Once the optimum $|S_{22}(j\omega)|$ and $|S_{12}(j\omega)|$ have been found from the integral constraints and Equation 75, it is always possible to find the complex rational functions S_{22} and S_{12} . These in conjunction with $S_{11} = 0$ specify the complete network which may then be synthesized according to methods already described.

It is interesting to compare the performance of optimum dissipative matching networks with optimum lossless networks (the latter designed on the basis of minimum input reflection amplitude over a prescribed band). This is easily done since the limiting factor in both lossy and lossless networks is the amplitude function $|S_{22}(j\omega)|$ as determined by the integral constraints imposed by the prescribed load. The minimum value of input reflection factor amplitude over a given pass band when lossless matching networks are used is $|S_{22}|_{\text{MIN}}$, and the maximum insertion gain of the lossless matching network when excited by a matched generator is:

$$P(\text{MAX. LOSSLESS}) = 1 - |S_{22}|_{\text{MIN}}^2$$

Equation (b) of Theorem 5 gives the maximum insertion gain of a dissipative matching network in terms of $|S_{22}|_{\text{MIN}}$ when $|S_{11}|$ is prescribed

$$P(\text{MAX. LOSSY}) = (1 + |S_{11}|)(1 - |S_{22}|)$$

The ratio of maximum gains when a given load is matched by optimum lossy and lossless networks is:

$$\frac{P(\text{MAX. LOSSY})}{P(\text{MAX. LOSSLESS})} = \frac{1 + |S_{11}|}{1 + |S_{22}|_{\text{MIN}}} \quad (76)$$

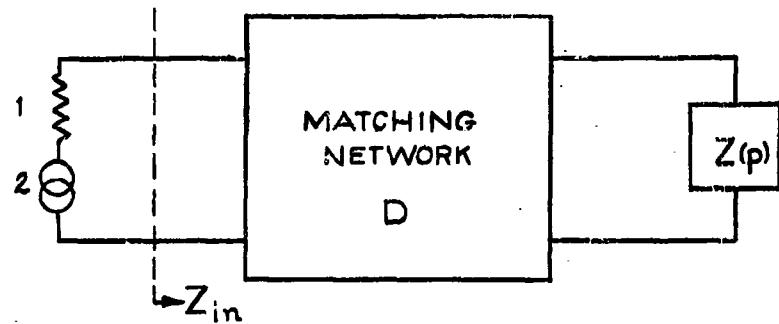
In the limit when $|S_{22}|_{\text{MIN}} = 1$, this becomes

$$\frac{P(\text{MAX. LOSSY})}{P(\text{MAX. LOSSLESS})} \Big|_{|S_{22}|_{\text{MIN}} \rightarrow 1} = \frac{1}{2} (1 + |S_{11}|) \quad (77)$$

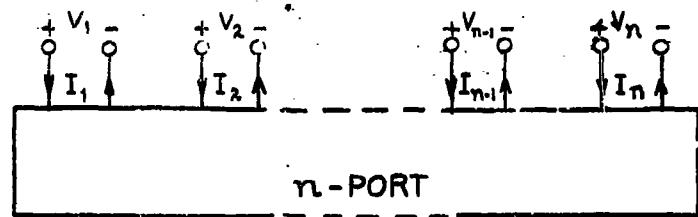
In the case of a reflectionless matching network $|S_{11}| = 0$, and the rather startling fact emerges that in the pass band the insertion loss of an optimum dissipative matching network which matches a load to a real generator impedance without reflections over the infinite real frequency band can never exceed that of an optimum lossless network by more than 3db. It should be pointed out that in practical problems $|S_{22}|_{\text{MIN}} < 1$, so that the difference in insertion loss between lossy and lossless matching is invariably considerably less than 3db.

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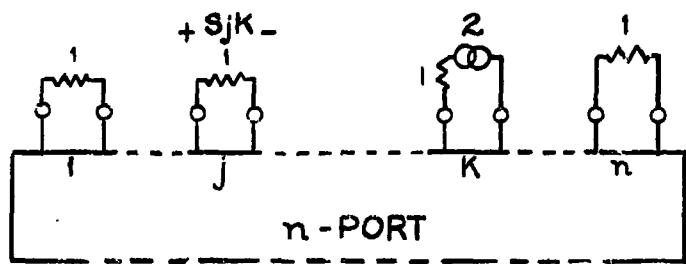
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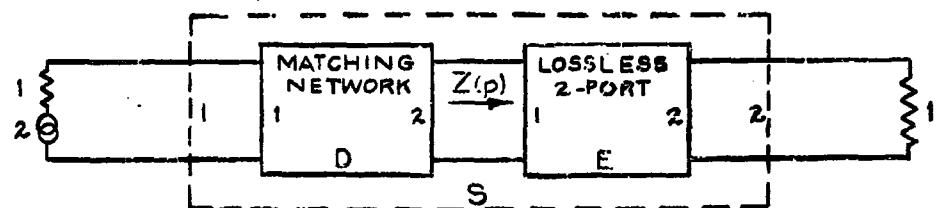
GENERATOR, MATCHING NETWORK, AND LOAD



GENERAL n -PORT

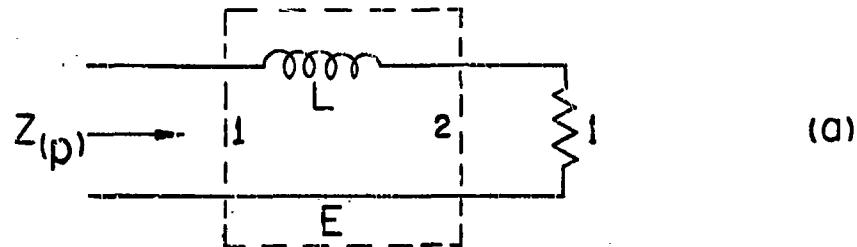


GENERAL n -PORT

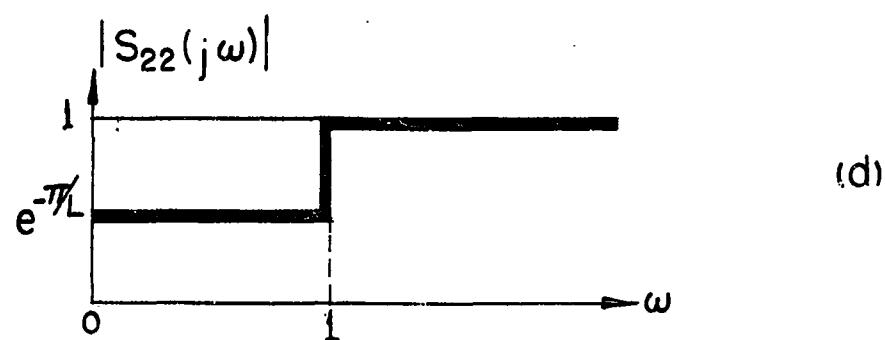
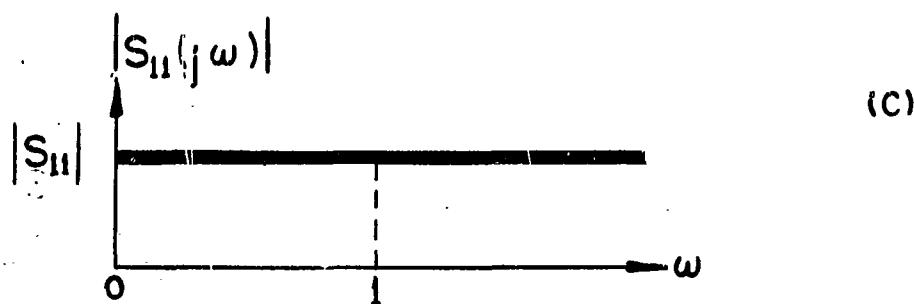
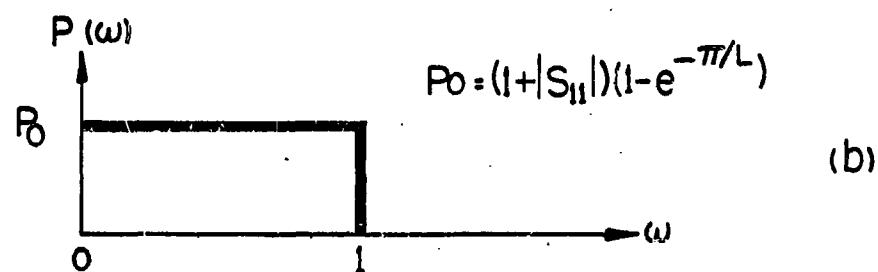


USE OF DARLINGTON REPRESENTATION OF LOAD $Z(p)$

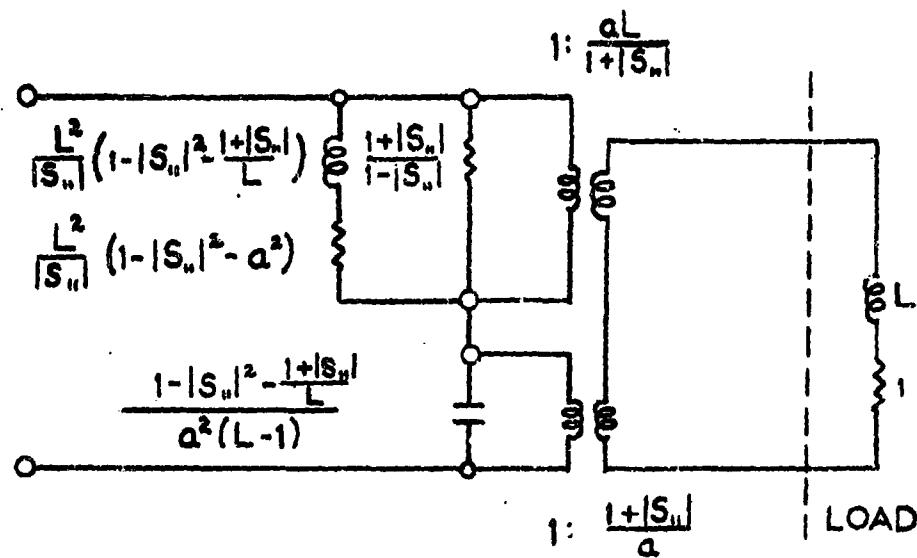
NETWORK CONFIGURATIONS & DEFINITIONS



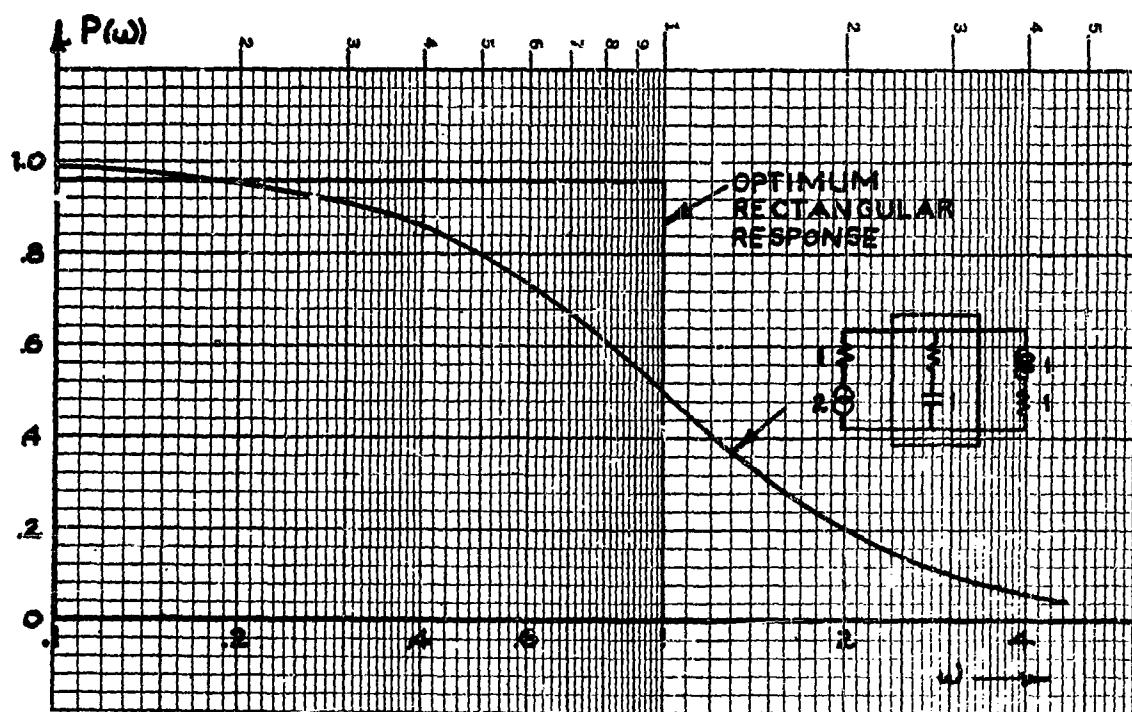
SIMPLE LOAD IMPEDANCE



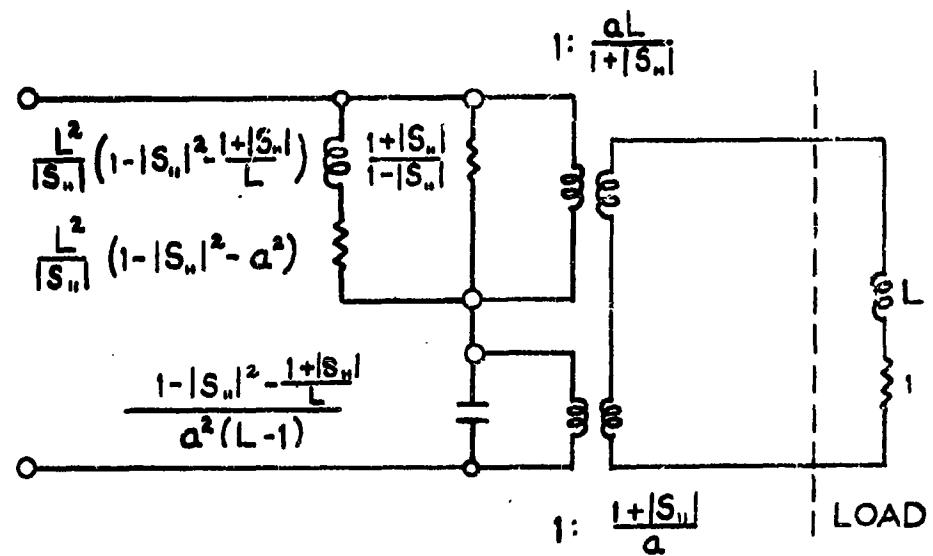
SIMPLE LOAD IMPEDANCE AND THEORETICALLY POSSIBLE PERFORMANCE CURVE



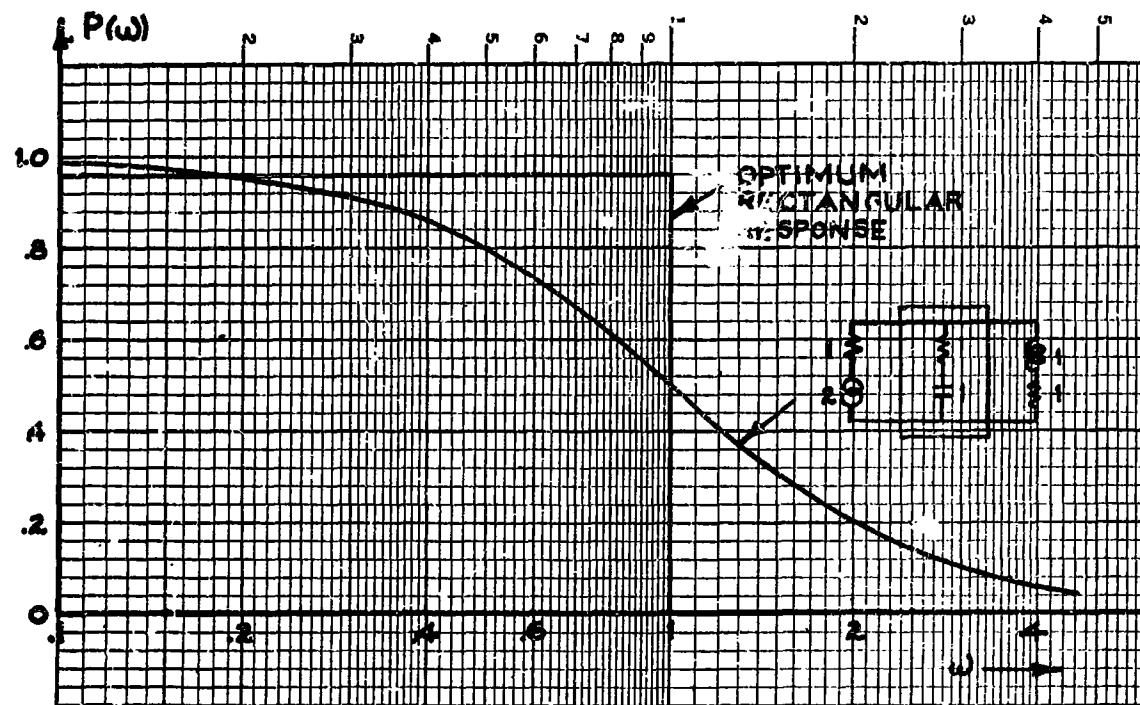
a. MATCHING NETWORK FOR SIMPLE LOAD



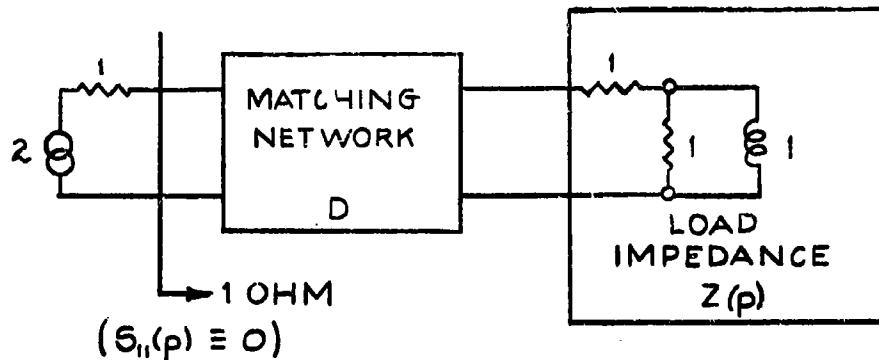
b. PERFORMANCE OF SIMPLE MATCHING NETWORK
COMPARED TO BEST POSSIBLE LOW-PASS RESPONSE



a. MATCHING NETWORK FOR SIMPLE LOAD

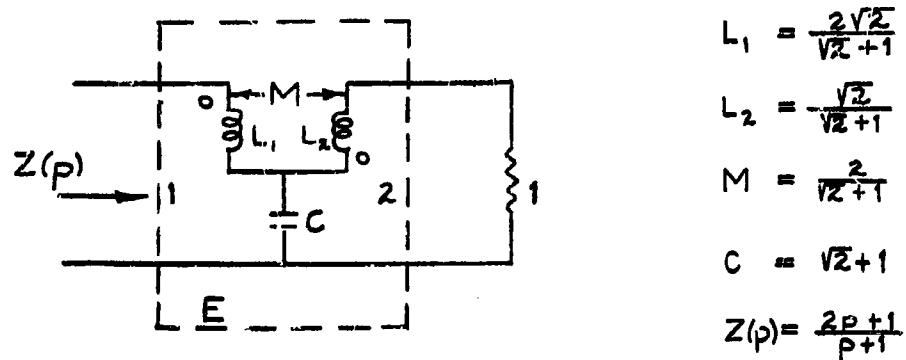


b. PERFORMANCE OF SIMPLE MATCHING NETWORK
COMPARED TO BEST POSSIBLE LOW-PASS RESPONSE



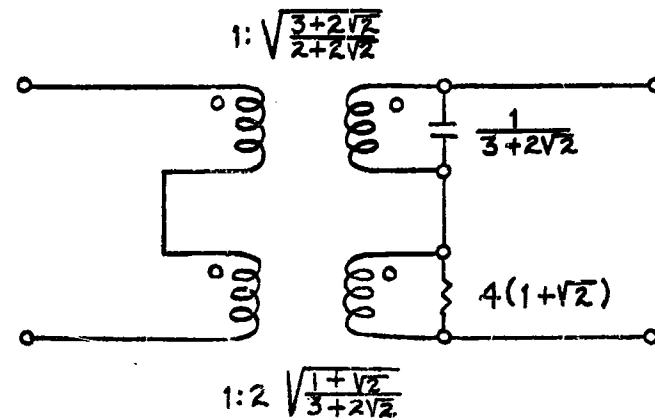
A.

MATCHING NETWORK AND SIMPLE LOAD



B.

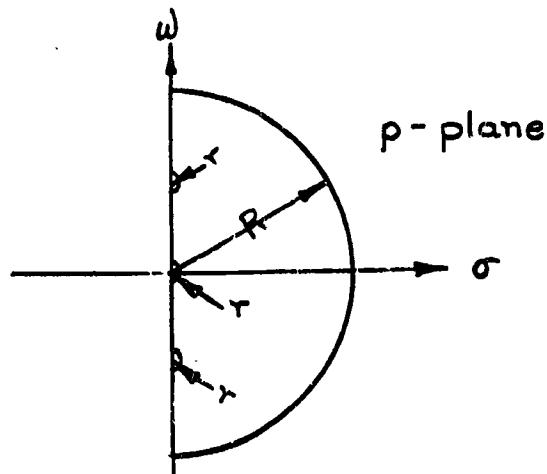
DARLINGTON REPRESENTATION OF $Z(p)$



C.

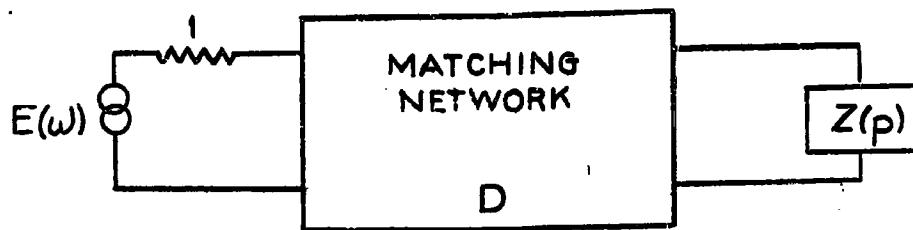
MATCHING NETWORK

SIMPLE LOAD AND MATCHING NETWORK

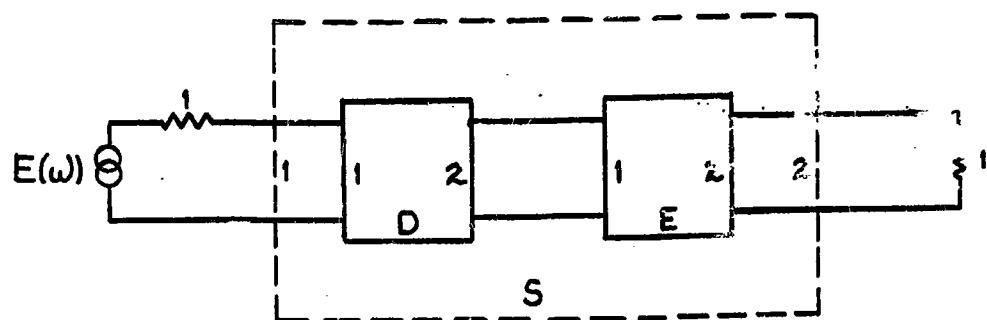


P- PLANE COUNTER

a.

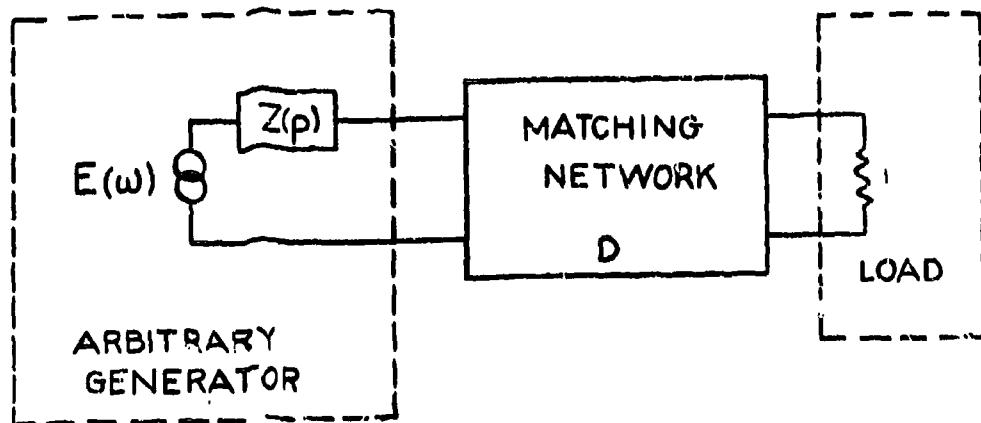


b.

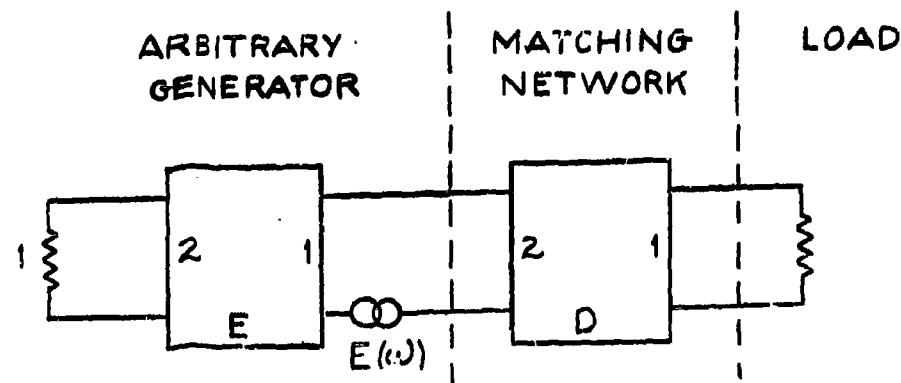


c.

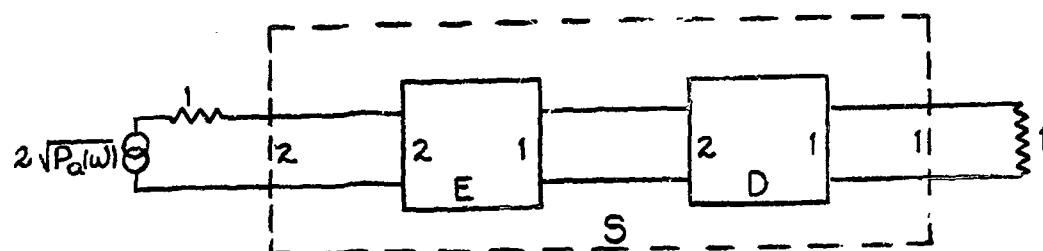
NORMALIZATION OF AVAILABLE
POWER OF GENERATOR



a.



b.



c.

MATCHING ARBITRARY GENERATOR
TO RESISTIVE LOAD